Time Value of Money Review Notes:

Be sure that you did the assigned reading from the first week of the course (sections 5.1-5.3 and 6.1-6.2) before reading this note. This review expands on the formulas introduced in your textbook to account for different compounding intervals. As part of this review I provide a number of examples. Be sure that as you read the review that you understand each example. If something is not clear please come by my office on a Monday or a Wednesday. The intuition from these formulas, if not the formulas themselves, should already be familiar to you from MBA 520 and last week's lecture.

By the end of this review, you should be familiar with the 4 equations that do the following 4 tasks:

1. Discount a future value to a present value. This formula is useful in moving a single cash flow backwards in time.
2. Convert a present value to a future value. This formula is useful in moving a single cash flow forward in time.
3. Calculate the present value or payment associated with an annuity. This formula is useful in valuing a finite stream of cash flows.
4. Calculate the present value or payment associated with a perpetuity. This formula is useful in valuing an infinite stream of cash flows.

(1) To discount a future value (FV) to a present value (PV) use the following formula:

$$PV = \frac{FV}{(1+r)^n}$$

where r is the effective annual discount rate (EAR) and n is the number of years. If you are dealing with compound interest and APRs then the formula needs to be adjusted to account for this. For example, if you had m compounding periods per year then in the formula you would first convert the APR to the effective rate over one compounding interval (i.e., divide APR by m) and then raise this effective rate to the "m*n" which represents the total number of compounding intervals in n years. So the formula would be written as follows:

$$PV = \frac{FV}{\left[1+\frac{APR}{m}\right]^{n*m}}$$

Note that $\frac{APR}{m}$ is the effective rate over 1 compounding interval and n*m is the total number of compounding intervals in n years.

Example 1: Let's say you were asked whether you preferred \$100 today or \$110 next year and that the appropriate discount rate was 6% (assume that 6% is an effective rate because no compounding information is provided). How would you answer this question? You would calculate the present value of the $110 and then compare that with the $100 (which is already in today's dollars). The present value of the $110 would be

$$\mathrm{PV} = \frac{\mathrm{FV}}{(1+\mathrm{r})^{\mathrm{n}}} = \frac{110}{(1+.06)^{1}} = \$103.77$$

Example 2: What is the PV of $1,000 to be received 5 years from now if the discount rate is 10% APR, semi-annually compounded? To answer this question use the formula

$$\mathrm{PV} = \frac{\mathrm{FV}}{\left[1+\frac{\mathrm{APR}}{\mathrm{m}}\right]^{\mathrm{n*m}}} = \frac{100}{\left(1+\frac{.10}{2}\right)^{2*5}} = \$613.91.$$

(2) To convert a present value to a future value use the following formula:

$$\mathrm{FV} = \mathrm{PV}(1+r)^{n}$$

where r is the effective annual rate and n is the number of years. If you are dealing with compound interest and APRs then the formula needs to be adjusted to account for this. For example, if you had m compounding periods per year then in the formula you would first convert the APR to the effective rate over one compounding interval (i.e., divide APR by m) and then raise this effective rate to the "m*n" which represents the total number of compounding intervals in n years. So the formula would be written as follows:

$$FV = PV\left(1+\frac{APR}{m}\right)^{n*m}$$

Note that $\frac{\mathrm{APR}}{\mathrm{m}}$ is the effective rate over 1 compounding interval and n*m is the total number of compounding intervals in n years.

Example 1: How much would $100 be worth in 50 years with 6% annual interest? The answer to this question is the future value of $100 after it experiences 50 years of interest. The formula would be $\mathrm{FV} = \mathrm{PV}(1+r)^{n} = 100(1.06)^{50} = \$1{,}842.015$.

Example 2: You just inherited $50,000. You are interested in setting the $50,000 aside to help your daughter with future college costs. Assume your daughter is 1 years old today and that she will go to college at the age of 18. If you can invest the money and earn 6% APR compounded monthly, how much money will you have when your child turns 18? To answer this question use the formula $FV = PV\left(1+\frac{APR}{m}\right)^{n*m} = 50{,}000\left(1+\frac{.06}{12}\right)^{17*12} = \$138{,}307.80$.

(3) To calculate the present value of an annuity that pays a fixed payment (PMT) over time use the following formula:

$$PV\ of\ annuity = PMT\left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right]$$

where r is the effective annual rate and n is the number of years over which you receive an annual payment of PMT. If you are dealing with compound interest and APRs then the formula needs to be adjusted to account for this. The adjustment is a little more complicated for an annuity. To see why, assume for a moment that you made the same adjustment as in (1) and (2) above with m compounding periods per year and so in the formula you converted the APR to the effective rate over one compounding interval (i.e., divide APR by m) and then raised this effective rate to the "m*n". Doing this would result in the following formula:

$$PV\ of\ annuity = PMT\left[\frac{1}{\frac{APR}{m}} - \frac{1}{\left(\frac{APR}{m}\right)\left(1+\frac{APR}{m}\right)^{nm}}\right]$$

The problem with this formula is that it is for n*m payments and not n payments. So, for example, if you had monthly compounding *and* monthly payments (m=12) then you could use the above equation. But as stated above we have annual payments but sub-annual compounding. For our situation you need to convert the rate to an EAR and then raise it to "n" instead of "n*m".

So first find the EAR: EAR $=\left(1+\frac{APR}{m}\right)^m - 1$
Then use the original PV of an annuity formula with r being the effective annual rate.

$$PV\ of\ annuity = PMT\left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right]$$

Example 1: Assume you own a 10% coupon bond that has a par value of $1000 and pays an annual coupon. The bond has 10-years until maturity. Assume the discount rate is 6%. What is the present value of the stream of coupon payments (not including the final $1000 payment)? What is the price of the bond?

To answer the first question use the formula

$$PV = PMT\left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right] = 100\left[\frac{1}{.06} - \frac{1}{.06(1+.06)^{10}}\right] = \$736.008$$

So the present value of a 10-year annuity of coupon payments in this example is worth $736.01. To find the price of the bond we need to then add the present value of the final $1000 payment that will occur 10 years in the future to the present value of the coupon payments. To calculate the present value of this single cash flow at the end of the 10 years we use the formula

$$PV = \frac{FV}{(1+r)^n} = \frac{1000}{(1+.06)^{10}} = \$558.395$$

So the price of the bond equals the present value of the coupon payments plus the present value of the final $1000 payment ($736.01+$558.395).

Example 2: Congratulations! You just won the lottery… but you have to choose between 3 possible payment options (all values are after-tax):

1) Get a lump sum of $38 million today
2) Get paid in 30 annual NOMINAL installments of $3 million, starting today
3) Get paid in 30 annual REAL installments of $2 million, starting today. In this context, a "real" payment simply means that the future payments will have the same purchasing power as $2 million does today.

For this problem assume your nominal discount rate is 7%, compounded monthly and that inflation is 3%, compounded monthly. Which option should you choose?

Here's a timeline:

	Year 0 (today)	1	2	3	…	28	29	30
Option 1	38							
Option 2 (nominal CFs)	3	3	3	3	…	3	3	0
Option 3 (real CFs)	2	2	2	2	…	2	2	0

We know that the PV of the first option is $38 million. We need to calculate the PV of the other 2 options in order to compare the different payout options.

The 2nd payment option is a 30-year "annuity due" which means the first payment occurs today as opposed to one year in the future. Another way to think about the 2nd payment option is that it is a standard 29-year annuity with an additional payment added on today. In a normal annuity the first payment is 1 period (1 year in this example) in the future. In an "annuity due" the first payment occurs today.

For the 2^{nd} payout option we use the nominal discount rate to discount the nominal cash flows.

Even though the rates are compounded at a monthly level the payments are made at an annual level. This is important to note because it would be wrong to use the annuity formula with an exponent (m*n) = (12*29) in this case because this would suggest monthly payments over 29 years when in fact the payments are annual. To get around this we need to convert the rates to effective annual rates as follows:

$$EAR = \left(1 + \frac{APR}{m}\right)^{m} - 1$$

$$EAR_{nom} = \left(1 + \frac{0.07}{12}\right)^{12} - 1 = 0.07229$$

So the PV of the 2nd option = today's payment + the PV of a 29 year annuity discounted using an effective nominal rate of 7.229%.

$$PV = 3 + 3\left(\frac{1}{0.07229} - \frac{1}{0.07229(1.072229)^{29}}\right) = \$39.017 \text{ M}$$

For the 3rd payment option, we need to calculate the PV of the stream of annual real cash flows using a real rate. The 3rd payment option is another 30-year "annuity due" which is the same as a standard 29-year annuity plus today's payment. We need to use an effective annual real discount rate instead of a nominal discount rate to discount annual real cash flows. Using the Fisher equation we know that (1+nominal rate) divided by (1+ inflation rate) equals (1+ real rate). So the effective annual real rate in this problem would be

$$EAR_{\inf} = \left(1 + \frac{0.03}{12}\right)^{12} - 1 = 0.03042 \quad \leftarrow \text{Effective annual inflation}$$

$$1 + EAR_{real} = \frac{1 + EAR_{nom}}{1 + EAR_{\inf}} = \frac{1.07229}{1.03042} = 1.0406 \quad \leftarrow \text{1+ effective real rate}$$

$$EAR_{real} = 0.0406$$

So the PV of the 3rd option = today's payment + the PV of a 29 year annuity discounted using an effective real rate of 4.06%.

$$PV = 2 + 2\left(\frac{1}{0.0406} - \frac{1}{0.0406(1.0406)^{29}}\right) = \$35.727$$

So in comparing the 3 options the comparison is between 38M, 39.02M, and 35.72M. The second is the best option.

(4) To calculate the present value of a perpetuity that pays a fixed payment (pmt) forever use the following formula:

$$PV\ of\ perpetuity = \frac{PMT}{r}$$